K-STABILITY OF FANO WEIGHTED HYPERSURFACES

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joint with
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Main characters: Fano n-fold hypersurfaces.

$\mathbb{Q}$-Fano n-folds: normal, projective n-dimensional, $-K_X$ ample, with $\mathbb{Q}$-factorial terminal singularities.

$$X = X_d \subset w\mathbb{P}^{n+1} = \mathbb{P}(a_0, \dots, a_{n+1})$$

$X_d = (F=0)$, $\deg F = d$

where:
- $a_0 \leq \dots \leq a_{n+1} \in \mathbb{Z}_{>0}$.
- $w\mathbb{P}^{n+1}$ is well-formed, i.e. $\gcd(a_0, \dots, \hat{a}_i, \dots, a_{n+1}) = 1 \ \forall i$.

and X is:
- well-formed, i.e. $w\mathbb{P}^{n+1}$ is well-formed and $\mathrm{Sing}\,\mathbb{P} \cap X \subset X$ has codim ≥ 2
- quasismooth, i.e. fixing coords $x_0, \dots x_{n+1}$, $\mathrm{wt}(x_i) = a_i$ and for $I \subset \mathbb{C}[x_0, \dots, x_{n+1}]$ homogeneous ideal s.t. $R := \mathbb{C}[x_0, \dots, x_{n+1}]/I$ and $X = \mathrm{Proj}\, R$

  define $C_X := \mathrm{Spec}\, \mathbb{C}[x_0 \dots, x_{n+1}]/I$ affine scheme

  $C_X^* := C_X \setminus \{0\}$

  X is quasismooth if C_X^* is smooth.

- Fano index 1, i.e. $-K_X \sim \iota_X A$, $A \in \mathrm{Cl}(X)$ ample class and $1 = \iota_X = \sum_{i=0}^{n+1} a_i - d$

WANT TO STUDY: K-stability of these hypersurfaces

What do we know about:

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$m=3$ there are 95 distinguished families [Reid, Iano-Fletcher] [Chen, Chen, Chen]

The picture is complete:

general case: [Cheltsov]

- **Theorem:** [Kim, Okada, Won] (α-inv), $\overbrace{\text{[Sano, Tasin]}}^{\delta\text{-inv}}$, [Campo, Okada]

$$\boxed{\text{All quasismooth members of the 95 families of Fano 3-fold hypersurfaces are K-stable.}}$$

Topic of this talk:

$m>3$ First result:

- **Theorem:** [Sano, Tasin]

$X = X_d \subset w\mathbb{P}^{m+1}$ well-formed quasismooth hypersurface, $\iota_X = 1$. Assume a_{m+1} / d.

$\Longrightarrow \delta(X) > 1$, so it is K-stable

↳ This amounts to assuming that the singularity of highest order at $P_{m+1} = [0, \dots, 0, 1] \notin X$

in our joint work we address this assumption

Our **main theorems** are the following:

- **Theorem 1:** [CFST]

Let $a > 1$, $m \geq 3$ be integers.

$X = X_d \subset \mathbb{P}(1^{m+1}, a)$ quasismooth Fano weighted hypersurface of deg d, $\dim = m$, $\iota_X = 1$.

Then, $\delta(X; \mathcal{O}_X(1)) \geq \frac{m+1}{m} > 1$.

In particular, X is K-stable.

- Theorem 2 : [CFST]

Let $2 \le a \le b$, $m \geq 3$ be integers.

$X = X_d \subset \mathbb{P}(1^m, a, b)$ quasismooth Fano weighted hypersurface of deg d, $\dim = m$, $\iota_X = 1$.

Then, $\delta(X; \mathcal{O}_X(1)) \geq \dfrac{m+1}{m+\frac{1}{a}} > 1$

In particular, X is K-stable.

- Theorem 3 : [CFST]

Let $m \geq 3$, $c_1 \in \mathbb{Z}_{>0}$, $c_1 \geq \frac{m+2}{2}$ be integers, and $2 \le a_{c_1} \le \dots \le a_{m+1}$.

$X = X_d \subset \mathbb{P} = \mathbb{P}(1^{c_1}, a_{c_1}, \dots, a_{m+1})$ quasismooth Fano weighted hypersurface of deg d, $\dim = m$, $\iota_X = 1$.

Assume: • $d \equiv 1 \mod a_i$ $\forall\, c_1 \le i \le m+1$

• X_d general in the linear system $|\mathcal{O}_{\mathbb{P}}(d)|$

Then, X is K-stable

Tools we use to prove Theorems 1, 2, 3:

a combination of:
- Abban-Zhuang Method
- weighted blow-ups
- coverings
- Okounkov bodies

Let's start with defining δ-invariants, and how they measure K-stability

Take X a Klt Fano, and E a prime divisor over X, ie either $E \subset X$ or $E \subset V$ where $f: V \longrightarrow X$ is some birational model of X.

Let's define:

$A_X(E) := 1 + \mathrm{ord}_E(K_V - f^* K_X)$ log discrepancy

$$S_X(E) := \frac{1}{(-K_X)^n} \int_0^{+\infty} \mathrm{vol}(-f^*K_X - tE)\, dt$$ <u>Fujita-Li invariant</u>

Here, $\mathrm{vol}(D) := \lim_{m \to +\infty} \frac{h^0(mD)}{m^n/n!}$

- D nef $\Rightarrow$ $\mathrm{vol}(D) = D^n$
- D big $\Leftrightarrow$ $\mathrm{vol}(D) > 0$

So, $\mathrm{vol}(-f^*K_X - tE) > 0$ for $-f^*K_X - tE$ big,
and $\neq 0$ as long as $-f^*K_X - tE$ is pseudoeffective

$\Rightarrow$ $S_X(E)$ is actually a definite integral in the interval $[0, \tau]$ for $\tau := \max_t \{ -f^*K_X - tE \text{ is pseff} \}$
<u>pseudoeffective threshold</u>

$\rightsquigarrow$ $\delta_X(E) := \frac{A_X(E)}{S_X(E)}$,

and $\boxed{\delta(X) := \inf_{\substack{E/X \\ \text{prime}}} \delta_X(E)}$ global δ-invariant

$\rightarrow$ also, notation: $\delta(X; -K_X)$

$\rightsquigarrow$ as a consequence of the valuative criterion

$$\boxed{X \text{ is K-stable} \Leftrightarrow \delta(X) > 1}$$

$\uparrow$ so that's what we need to show

Moreover, we can define for $p \in X$

$$\boxed{\delta_p(X) := \inf_{\substack{E/X \\ \text{prime} \\ p \in f(E)}} \delta_X(E)}$$ <u>local δ-invariant at p</u>

TOOL 1: ABBAN-ZHUANG METHOD & WEIGHTED BLOWUPS

Rough idea: consider a flag $Y_\bullet$: $X = Y_0 \supset Y_1 \supset \dots \supset Y_j$ $j \leq n$

of subvarieties in X satisfying certain conditions on their singularities.

Then, it is possible to define a local δ-invariant wrt the flag, and to produce an estimate of the local δ-invariant.

In truth, these flags are plt flags, i.e. they have good singularities (i.e. purely log terminal) and each piece of the flag might actually sit inside a plt (weighted) blow-up of X.

$\Rightarrow$ we either have an honest flag $X \supset Y_1 \supset \dots \supset Y_j$

or we have

$$\begin{array}{c} \tilde{Y}_2 \supset \tilde{Y}_3 \ \dots \\ \Big\downarrow \text{blow-up} \\ X \supset Y_1 \supset Y_2 \end{array}$$

The estimate wrt $Y_\bullet$ is "recursive" as follows.

- Lemma: [Abban-Zhuang]

$p \in X$ point. Δ effective $\mathbb{Q}$-Weil divisor on X.

L big $\mathbb{Q}$-line bundle on X. $\alpha \in \mathbb{Q}_{>0}$.

$Y \in |\alpha L|$ irreducible & reduced divisor s.t. $p \in Y$ and the pair $(X, \Delta + Y)$ is plt at p.

Take Δ_Y an effective $\mathbb{Q}$-Weil divisor on Y s.t.
$(K_X + \Delta + Y)|_Y = K_Y + \Delta_Y$ holds around p.

$$\Rightarrow \quad \delta_p(X, \Delta; L) \geq \min\left\{ \alpha(n+1), \frac{n+1}{n} \delta_p(Y, \Delta_Y; L|_Y) \right\}$$

Tool 2: Coverings

When we have a finite cover $\tau : X' \to X$, we can dominate the δ-invariant at $p \in X$ with the δ-invariant at $p' \in X'$, $\tau(p') = p$.

- Proposition:

> (X, Δ), (X', Δ') pairs of normal projective varieties with effective $\mathbb{Q}$-Weil divisors.
>
> $\tau : X' \to X$ s.t. $K_{X'} + \Delta' = \tau^*(K_X + \Delta)$
>
> $p \in X$, $p' \in X'$ s.t. $\tau(p') = p$ and (X, Δ) is klt at p
>
> Take L a big $\mathbb{Q}$-line bundle on X.
>
> $\Rightarrow \quad \delta_{p'}(X', \Delta'; \tau^* L) \leq \delta_p(X, \Delta; L)$

$\hookrightarrow$ so, if we can say that $\delta_{p'}(X') > 1$, we're good.

Tool 3: Okounkov bodies

To a flag $Y_\bullet$ of X and L a big $\mathbb{Q}$-line bundle on X we can associate a convex object $\Delta_{Y_\bullet}(L)$.

Then:

- Theorem: [Fujita]

> $S_X(L; Y_1 \supset \dots \supset Y_j) =$ j-th coordinate of the barycentre of $\Delta_{Y_\bullet}(L)$.

$\hookrightarrow$ so we use Okounkov bodies to effectively compute Fujita-Li invariants

GENERAL STRATEGY TO PROVE Thm 1 & Thm 2

Construct flags by cutting X with an $H \in |\mathcal{O}_{\mathbb{P}}(1)|$.

Combine Abban-Zhuang and coverings to reduce the problem of
estimating δ-invariants at points $p \in X$
to doing so on quasismooth surfaces.

Once there, we resort to Okounkov bodies to compute the estimates.

__NB__: for $p = P_{n+1}$, we start with a weighted blowup

⚠ Technical caveat: need to make sure that the cut
we perform is always quasismooth.

Let's look at Thm 1:

We have $\quad X_d \subset \mathbb{P}(1^{n+1}, a) \qquad a > 1, \ d > 1$

$$P_{n+1} := [0, \dots 0, 1] \in X$$

Set $X = X_{ak+1} \subset \mathbb{P}(1^{n+1}, a)$

$\ \ \ \ \ ''$
$(F=0)$

$$F = x_{n+1}^{k} x_0 + x_{n+1}^{k-1} f_{a+1}(x_0, \dots, x_n) + \dots + f_{ak+1}(x_0, \dots, x_n)$$

and define $m := \min \{ t \in \mathbb{Z} \cap [1, k] \mid f_{at+1} \text{ is } \underline{\text{not}} \text{ divisible by } x_0 \}$

__$p \neq P_{n+1}$, ie smooth__ :
$$\delta_p(X, \mathcal{O}_X(1)) \geqslant \frac{(n+1)a}{d}$$

__$p = P_{n+1}$__ : distinguish two cases:

- $m \leq k-1$ Then we cut until a surface $S \subset \mathbb{P}(111a)$
  and we use Okounkov bodies.

- $m = k$ i.e. X has a Generalised Eckardt point at P_{n+1}
  $\Rightarrow$ its equation is:

$$F = x_{m+1} x_0 g + f_{ak+1}(x_0, \dots, x_m)$$

$$g := x_{m+1}^{k-1} + x_{m+1}^{k-2} g_a(x_0, \dots, x_m) + g_{a(k-1)}(x_0, \dots, x_m)$$

In this context, we find:

K-UNSTABLE HYPERSURFACES:

In the Eckardt case:

$a \geq 2$, $\quad m > \dfrac{a^2 k(k-1)}{a-1} \quad \Rightarrow \quad X$ is a K-unstable Klt Fano

Proof: (Sketch) $\quad X = X_{ak+1} \subset \mathbb{P}(1^{m+1}, a)$

$$\Rightarrow -K_X \sim \mathcal{O}_X(m+1+a-(ak+1)) = \mathcal{O}_X(\underbrace{m+a-ak}_{\substack{\vee \\ 0}})$$

$\because$ assumption of Fano

In the Eckardt case, we can show that

$$\delta(X) = \frac{1}{m+a-ak}\, \delta(X; \mathcal{O}_X(1)) \geq \frac{m(m+1)}{(m+a-ak)(ak+m)}$$

And when is $\delta(X) > 1$?

when $(m+a-ak)(ak+m) - m(m+1) < 0$

$\Rightarrow$ stability only when $m < \dfrac{a^2 k(k-1)}{a-1}$

($=$ is semistability) $\square$

Now, $\iota_X = m + a - ak = m - (k-1)a$.

$$\Rightarrow \left(\text{using } m > \frac{a^2 k(k-1)}{a-1}\right) \quad \iota_X > \frac{a^2 k(k-1)}{a-1} - (k-1)a =$$

$$= \frac{((k-1)a+1)(k-1)a}{a-1}$$

So for instance, for $k=2$ we have

$$\iota_X > \frac{a(a+1)}{a-1} \geq 6 \quad \because a \geq 2.$$

$\Rightarrow \forall a \geq 2, \forall m \in \mathbb{Z}_{>0}$ s.t. $m > \dfrac{2a^2}{a-1}$

we find a whole class of Fano n-fold hypersurfaces

$X_{2a+1} \subset \mathbb{P}(1^{n+1}, a)$ $\quad i_X = n-a$

that are K-unstable.